A. Matrices and Types of Matrices

1. Definition of a Matrix

A **matrix** is defined as an ordered rectangular array of numbers or functions. The numbers or functions within this array are called the **elements** or **entries** of the matrix. Matrices are typically denoted by capital letters (A, B, C).

A matrix with m rows and n columns is said to have the **order** $m \times n$ (read as "m by n"). The general representation of an element lying in the i^{th} row and j^{th} column is a_{ij}.

2. Kinds of Matrices

The sources identify several specific types of matrices based on their structure:

- **Column Matrix:** A matrix having only one column. Its general order is $m \times 1$.
- **Row Matrix:** A matrix having only one row. Its general order is $1 \times n$.
- **Square Matrix:** A matrix where the number of rows equals the number of columns ($m = n$). For a square matrix of order n, the elements $a_{11}, a_{22}, \ldots, a_{nn}$ constitute its **diagonal**.
- **Diagonal Matrix:** A square matrix where all non-diagonal elements are zero ($a_{ij} = 0$ when $i \neq j$).
- **Scalar Matrix:** A diagonal matrix where all diagonal elements are equal to a constant k.
- **Identity Matrix:** A square matrix where all diagonal elements are 1 and all non-diagonal elements are 0. It is denoted by I_n or simply I.
- **Zero (Null) Matrix:** A matrix where all elements are zero. It is denoted by O.

B. Equality, Transpose, and Symmetry

1. Equality of Matrices

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal** ($A = B$) if and only if:

1. They are of the same order.
2. Each element of A is equal to its corresponding element in B ($a_{ij} = b_{ij}$ for all i, j).

2. Transpose of a Matrix

The **transpose** of a matrix A (denoted by A' or A^T) is obtained by interchanging the rows and columns of A. If A is of order $m \times n$, then A' is of order $n \times m$.

Properties of Transpose:

- $(A')' = A$
- $(kA)' = kA'$ (where k is a scalar)
- $(A+B)' = A' + B'$
- $(AB)' = B'A'$ (Reversal Law)

3. Symmetric and Skew-Symmetric Matrices

- **Symmetric Matrix:** A square matrix A is symmetric if $A' = A$. This means $a_{ij} = a_{ji}$ for all i, j.
- **Skew-Symmetric Matrix:** A square matrix A is skew-symmetric if $A' = -A$. This implies $a_{ji} = -a_{ij}$. Notably, for $i = j$, we get $a_{ii} = -a_{ii}$, which means **all diagonal elements of a skew-symmetric matrix must be zero**.

Key Theorems:

- For any square matrix A with real entries, $A + A'$ is always symmetric, and $A - A'$ is always skew-symmetric.
- Any square matrix can be uniquely expressed as the sum of a symmetric and a skew-symmetric matrix: $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$.

C. Algebra of Matrices

1. Addition and Subtraction

Two matrices can be added or subtracted only if they are of the **same order**. The sum/difference is a matrix obtained by adding/subtracting corresponding elements.

- **Commutative Law:** $A + B = B + A$.
- **Associative Law:** $(A + B) + C = A + (B + C)$.
- **Additive Identity:** The zero matrix O acts as the identity such that $A + O = A$.
- **Additive Inverse:** For every matrix A, there exists $-A$ such that $A + (-A) = O$.

2. Scalar Multiplication

Multiplying a matrix A by a scalar k results in a matrix where every element of A is multiplied by k.

- **Properties:** $k(A + B) = kA + kB$ and $(k + l)A = kA + lA$.

3. Matrix Multiplication

The product AB is defined only if the **number of columns in A equals the number of rows in B**. If A is $m \times n$ and B is $n \times p$, the resulting matrix $C = AB$ is of order $m \times p$. The entry c_{ik} is the sum of the products of elements of the i^{th} row of A with the k^{th} column of B: $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$

Properties of Multiplication:

- **Non-Commutative:** Generally, $AB \neq BA$.
- **Associative:** $(AB)C = A(BC)$.
- **Distributive:** $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$.
- **Multiplicative Identity:** For a square matrix A, $IA = AI = A$.
- **Zero Product:** The product of two non-zero matrices can be a zero matrix.

D. Determinant of Matrices

1. Definition and Expansion

To every square matrix A, we can associate a unique number called its **determinant**, denoted by $|A|$ or $\det A$.

- **Order 1:** If $A = [a]$, then $|A| = a$.
- **Order 2:** If $A = [a \quad b \; c \quad d]$, then $|A| = ad - bc$.
- **Order 3:** Expansion is done along any row or column. For row 1: $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$ where A_{ij} is the cofactor of element a_{ij}. Expanding along a row/column with the most zeros simplifies calculations.

2. Minors and Cofactors

- **Minor (M_{ij}):** The determinant of the submatrix obtained by deleting the i^{th} row and j^{th} column.
- **Cofactor (A_{ij}):** Defined as $A_{ij} = (-1)^{i+j} M_{ij}$.

3. Properties of Determinants

- **Singular Matrix:** A square matrix A is singular if $|A| = 0$.
- **Non-singular Matrix:** A square matrix A is non-singular if $|A| \neq 0$.
- **Product Rule:** For two square matrices of the same order, $|AB| = |A||B|$.
- **Scalar Property:** If A is of order n, then $|kA| = k^n|A|$.

E. Inverse of a Matrix

1. Definition and Invertibility

A square matrix A is **invertible** if there exists a square matrix B such that: $AB = BA = I$ In this case, B is the inverse of A ($B = A^{-1}$). A square matrix A is invertible **if and only if it is non-singular** ($|A| \neq 0$).

2. Finding Inverse Using Adjoint

The **adjoint** of a matrix A ($\text{adj } A$) is the transpose of the matrix of cofactors of A. The formula for the inverse is: $A^{-1} = \frac{1}{|A|}\text{adj } A$

Key Theorem: $A(\text{adj } A) = (\text{adj } A)A = |A|I$.

3. Properties of Inverse

If A and B are invertible square matrices of the same size:

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^{-1})^{-1} = A$
- $(A^T)^{-1} = (A^{-1})^T$
- $|\text{adj } A| = |A|^{n-1}$ (where n is the order)

F. Solving Systems of Simultaneous Equations

Matrices and determinants provide a systematic way to solve systems of n linear equations in n variables. A non-homogeneous system can be written as: $AX = B$ Where A is the coefficient matrix, X is the variable matrix, and B is the constant matrix.

1. Matrix Method (Unique Solution)

If A is non-singular ($|A| \neq 0$), the system has a **unique solution** given by: $X = A^{-1}B$

2. Consistency and Inconsistency

A system is **consistent** if it has one or more solutions and **inconsistent** if it has no solution.

- **Case** $|A| \neq 0$**:** Consistent, unique solution.
- **Case** $|A| = 0$**:**
  - If $(\text{adj } A)B \neq O$: The system is **inconsistent** (no solution).
  - If $(\text{adj } A)B = O$: The system may be either consistent (infinitely many solutions) or inconsistent.

Summary Table: Key Algebraic Relationships

Property	Formula
Inverse	$A^{-1} = \frac{1}{\det\ A}\text{adj}\ A$
Product Inverse	$(AB)^{-1} = B^{-1}A^{-1}$
Determinant Product	$$
System Solution	$X = A^{-1}B$
Adjoint Property	$$ A(\text{adj } A) =
Transpose Reversal	$(AB)^T = B^T A^T$