1. Continuity and Differentiability

A. Continuity

A real-valued function f is **continuous at a point** c in its domain if the limit of the function as x approaches c exists and is equal to the value of the function at c. Mathematically: $\lim_{x\to c} f(x) = f(c)$ For a function to be continuous, the **left-hand limit (LHL)**, the **right-hand limit (RHL)**, and the function value $f(c)$ must all coincide. A function is continuous on its domain if it is continuous at every point within that domain.

Algebra of Continuous Functions: If f and g are continuous at c, then their sum ($f+g$), difference ($f-g$), product ($f \cdot g$), and quotient (f/g, provided $g(c) \neq 0$) are also continuous at c. Notably, every polynomial, rational, and trigonometric function is continuous within its respective domain.

B. Differentiability

The **derivative** of a function f at point c is defined as the limit: $f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ If this limit exists, f is differentiable at c. A crucial theorem states that **every differentiable function is continuous**, but the converse is not true; for example, the modulus function $f(x) = |x|$ is continuous at $x = 0$ but not differentiable there.

C. Differentiation Rules and Techniques

- **Chain Rule:** Used to differentiate composite functions. If $f = v \circ u$ and $t = u(x)$, then: $\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$ This can be extended to compositions of three or more functions.
- **Implicit Functions:** When y is expressed implicitly as a function of x (e.g., $y + \sin y = \cos x$), we differentiate both sides with respect to x and solve for dy/dx.
- **Inverse Trigonometric Functions:** Standard derivatives include: $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$.
- **Exponential and Logarithmic Functions:** The exponential function $f(x) = e^x$ is unique because its derivative is itself. The derivative of $\log x$ is $1/x$.
- **Logarithmic Differentiation:** A powerful technique for functions of the form $f(x) = [u(x)]^{v(x)}$. By taking the natural log of both sides, the exponent is brought down, simplifying the differentiation process.
- **Parametric Forms:** If $x = f(t)$ and $y = g(t)$, the derivative is: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \text{ provided } \frac{dx}{dt} \neq 0$.
- **Second-Order Derivatives:** If $f'(x)$ is differentiable, its derivative is the second-order derivative, denoted as d^2y/dx^2 or $f''(x)$.

2. Applications of Derivatives

A. Rate of Change

The derivative dy/dx represents the **instantaneous rate of change** of y with respect to x. In real-world contexts, if s is distance and t is time, ds/dt represents velocity. If two variables x and y both vary with time t, their relative rate of change is $(dy/dt)/(dx/dt)$.

B. Increasing and Decreasing Functions

A function f is analyzed on an interval (a, b) based on its derivative:

- **Increasing:** If $f'(x) \geq 0$ for each x in the interval.

- **Decreasing:** If $f'(x) \leq 0$ for each x in the interval.
- **Strictly Increasing:** If $f'(x) > 0$.
- **Strictly Decreasing:** If $f'(x) < 0$.

C. Maxima and Minima

These are the "turning points" where a function reaches its highest or lowest values locally or globally.

- **Critical Points:** Points where $f'(c) = 0$ or f is not differentiable.
- **First Derivative Test:** If $f'(x)$ changes sign from positive to negative at c, it is a **local maxima**. If it changes from negative to positive, it is a **local minima**. If the sign does not change, it is a **point of inflection**.
- **Second Derivative Test:** If $f'(c) = 0$ and:
  - $f''(c) < 0$, then c is a point of **local maxima**.
  - $f''(c) > 0$, then c is a point of **local minima**.
- **Absolute Maxima/Minima:** On a **closed interval** $[a, b]$, the absolute maximum and minimum values are found by comparing the function values at all critical points and the endpoints a and b.

3. Integrals

Integration is the **inverse process of differentiation**. Given a function f, finding a function F such that $F'(x) = f(x)$ is called integration, and F is an **anti-derivative** or integral.

A. Methods of Integration

1. **Substitution:** Transforming the integral by substituting $x = g(t)$ to reduce it to a standard form.
2. **Partial Fractions:** Used for integrating rational functions $P(x)/Q(x)$ by decomposing them into simpler fractions.
3. **Integration by Parts:** Based on the product rule of differentiation: $$\int u \cdot v, dx = u \int v, dx - \int \left[u' \int v, dx\right] dx$$ A common rule for choosing the first function u is the **ILATE** priority (Inverse trig, Logarithmic, Algebraic, Trigonometric, Exponential).

B. Evaluation of Specific Integral Types

The sources provide standard formulae for various quadratic and irrational forms:

- $$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left|\frac{x-a}{x+a}\right| + C$$
- $$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log |x + \sqrt{x^2 \pm a^2}| + C$$
- $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C$$
- $$\int \sqrt{x^2 - a^2}, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log |x + \sqrt{x}$$
- $$\int \sqrt{a^2 - x^2}, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$$

For integrals of the type $\int \frac{dx}{ax^2 + bx + c}$, the denominator is transformed by **completing the square** to fit standard forms.

C. Definite Integrals

A definite integral $\int_a^b f(x)dx$ represents the **net area** between the curve and the x-axis.

- **First Fundamental Theorem of Calculus:** Defines the area function $A(x) = \int_a^x f(t)dt$ and states that $A'(x) = f(x)$.
- **Second Fundamental Theorem of Calculus:** Provides the practical method for evaluation: $\int_a^b f(x), dx = F(b) - F(a)$ where F is any anti-derivative of f.

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4. Applications of Integrals

The primary application is finding the **area under simple curves**.

- **Area Function:** The area of the region bounded by a curve $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by the definite integral $\int_a^b f(x)dx$.
- **Standard Curves:** This includes calculating areas for lines, circles, parabolas, and ellipses in their standard forms.

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5. Differential Equations

A **differential equation** involves an independent variable, a dependent variable, and its derivatives.

A. Basic Concepts

- **Order:** The order of the **highest-order derivative** present in the equation.
- **Degree:** The **power** of the highest-order derivative, provided the equation is a polynomial in its derivatives.
- **General Solution:** A solution containing as many arbitrary constants as the order of the equation.
- **Particular Solution:** Obtained by assigning specific values to the arbitrary constants in the general solution, usually based on given initial conditions.

B. Methods of Solution

1. **Separation of Variables:** If the equation can be expressed as $h(y)dy = g(x)dx$, it is solved by integrating both sides.
2. **Homogeneous Equations:** A first-order equation $dy/dx = F(x,y)$ is homogeneous if F is a homogeneous function of degree zero. These are solved by substituting $y = vx$.
3. **Linear Differential Equations:** Equations of the form: $\frac{dy}{dx} + Py = Q$ where P and Q are constants or functions of x only.
   - **Integrating Factor (I.F.):** $I.F. = e^{\int P,dx}$.
   - **General Solution:** $y(I.F.) = \int (Q \cdot I.F.), dx + C$.