Short Notes of Mathematics for CUET(B2)

UNIT III: CALCULUS

A. Higher Order Derivatives

1. Implicit and Parametric Differentiation

In standard differentiation, we often deal with explicit functions ($y = f(x)$). However, many relationships are more complex.

- **Implicit Functions:** These are relationships where y is not isolated on one side, such as $x - y = \pi$ or $y + \sin y = \cos x$. To find $\frac{dy}{dx}$, differentiate both sides of the equation with respect to x, treating y as a function of x and applying the **Chain Rule** where necessary.
- **Parametric Functions:** Sometimes x and y are both expressed in terms of a third variable, called a **parameter** (t or θ). For example, $x = f(t)$ and $y = g(t)$. The derivative is found using: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}, \text{ provided } f'(t) \neq 0$

2. Second and Higher Order Derivatives

If $y = f(x)$ is a differentiable function, its first derivative is $\frac{dy}{dx} = f'(x)$. If $f'(x)$ is also differentiable, its derivative with respect to x is called the **second order derivative**, denoted by $\frac{d^2y}{dx^2}$ or $f''(x)$.

- **Notation:** Higher orders are similarly defined. For the n^{th} order, we use $\frac{d^ny}{dx^n}$ or $y^{(n)}$.

B. Application of Derivatives

1. Rate of Change of Quantities

The derivative $\frac{dy}{dx}$ represents the **instantaneous rate of change** of y with respect to x.

- **Common Applications:**
  - **Velocity:** $\frac{ds}{dt}$ represents the rate of change of distance s with respect to time t.
  - **Geometric Change:** The rate of change of the area of a circle with respect to its radius r is $\frac{dA}{dr} = 2\pi r$.
- **Chain Rule in Rates:** If two variables x and y vary with respect to time t, then: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

C. Marginal Cost and Marginal Revenue

In economics, derivatives are used to find the "marginal" values of cost and revenue functions.

- **Marginal Cost (MC):** The instantaneous rate of change of total cost $C(x)$ with respect to the number of units x produced. $MC = \frac{dC}{dx}$
- **Marginal Revenue (MR):** The instantaneous rate of change of total revenue $R(x)$ with respect to the number of units x sold. $MR = \frac{dR}{dx}$

D. Increasing and Decreasing Functions

A function is analyzed based on the sign of its first derivative over an interval I.

- **Analytical Definitions:**
  - **Strictly Increasing:** If $x_1 < x_2$ implies $f(x_1) < f(x_2)$.
  - **Strictly Decreasing:** If $x_1 < x_2$ implies $f(x_1) > f(x_2)$.
- **First Derivative Test for Monotonicity:**
  - f is **increasing** if $f'(x) > 0$ for each x in the interval.

- f is **decreasing** if $f'(x) < 0$ for each x in the interval.
- f is **constant** if $f'(x) = 0$.

E. Maxima and Minima

1. Critical Points

A point c in the domain of f is a **critical point** if $f'(c) = 0$ or if f is not differentiable at c.

2. Local Maxima and Minima

- **First Derivative Test:**
  - **Local Maxima:** $f'(x)$ changes from positive to negative as x increases through c.
  - **Local Minima:** $f'(x)$ changes from negative to positive as x increases through c.
- **Second Derivative Test:** Let f be twice differentiable at c.
  - **Local Maxima:** If $f'(c) = 0$ and $f''(c) < 0$.
  - **Local Minima:** If $f'(c) = 0$ and $f''(c) > 0$.
  - **Failure:** If $f''(c) = 0$, the test is inconclusive; revert to the First Derivative Test.

3. Absolute Maxima and Minima

For a continuous function on a **closed interval** $[a, b]$, the **absolute maximum** and **absolute minimum** must exist.

- **Working Rule:**
  1. Find all critical points in (a, b).
  2. Evaluate $f(x)$ at these critical points and at endpoints a and b.
  3. The highest value is the absolute maximum; the lowest is the absolute minimum.

F. Integration: Indefinite Integrals

Integration is the **inverse process of differentiation**. Given a derivative $f(x)$, we seek an **anti-derivative** $F(x)$ such that $F'(x) = f(x)$.

- **The Arbitrary Constant:** Since the derivative of a constant is zero, an indefinite integral always includes a constant C.

$$\int f(x)dx = F(x) + C$$

1. Standard Integral Formulae

- $\int x^n dx = \frac{x^{n+1}}{n+1} + C, (n \neq -1)$
- $\int e^x dx = e^x + C$
- $\int \frac{1}{x} dx = \log|x| + C$
- $\int \cos x dx = \sin x + C$

G. Methods of Evaluating Indefinite Integrals

1. Integration by Substitution

Used to transform an integral by changing the variable x to t (where $x = g(t)$). It is effective when the derivative of a part of the integrand is also present.

$$\int f(g(x))g'(x)dx = \int f(t)dt, \text{ where } t = g(x)$$

2. Integration using Partial Fractions

Applied to **rational functions** $\frac{P(x)}{Q(x)}$.

- If the degree of $P(x) \geq Q(x)$, perform long division first.
- Decompose proper rational functions into sums of simpler fractions based on the factors of $Q(x)$.

3. Integration by Parts

Used for the **integral of the product** of two functions.

$$\int u \cdot v dx = u \int v dx - \int \left[\frac{du}{dx} \int v dx \right] dx$$

- **Rule of Thumb:** Use the **ILATE** priority (Inverse, Logarithmic, Algebraic, Trigonometric, Exponential) to choose the "first" function u.

H. Definite Integral and Area under Curve

1. Definition and Fundamental Theorem

A definite integral $\int_a^b f(x)dx$ has a unique value representing the **net signed area** between the curve and the x-axis from $x = a$ to $x = b$.

- **Second Fundamental Theorem:** $\int_a^b f(x)dx = F(b) - F(a)$ where F is an anti-derivative of f.

2. Properties of Definite Integrals

- **P4 (King's Property):** $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
- **P7 (Even/Odd):**
  - If $f(-x) = f(x)$ (Even), $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$.
  - If $f(-x) = -f(x)$ (Odd), $\int_{-a}^a f(x)dx = 0$.

3. Definite Integral as Area

The area A of a region bounded by $y = f(x)$, the x-axis, and lines $x = a, x = b$ is:

$$Area = \int_a^b |f(x)|dx$$

- If the curve is below the x-axis, the integral will be negative; take its **absolute value** for area.

I. Applications of Integration: Consumer and Producer Surplus

While the provided sources primarily focus on the geometric application of area, integration in economics is used to determine **Consumer Surplus (C.S.)** and **Producer Surplus (P.S.)**.

- **Note from outside the sources:** C.S. is the area between the demand curve and the price line, while P.S. is the area between the price line and the supply curve. You may want to independently verify the specific equilibrium formulas: $C.S. = \int_0^{x_0} f(x)dx - (p_0 \cdot x_0)$ $P.S. = (p_0 \cdot x_0) - \int_0^{x_0} g(x)dx$

J. Differential Equations: Basics

1. Recognition and Definitions

A **differential equation** involves an independent variable, a dependent variable, and its derivatives.

- **Order:** The order of the **highest order derivative** present in the equation.
- **Degree:** The **power** of the highest order derivative, provided the equation is a polynomial in derivatives.

K. Formulating and Solving Differential Equations

1. Solutions

- **General Solution:** A solution containing as many arbitrary constants as the order of the equation.
- **Particular Solution:** Obtained by giving specific values to the constants in the general solution.

2. Variable Separable Method

If an equation can be written as $h(y)dy = g(x)dx$, solve by integrating both sides: $\int h(y)dy = \int g(x)dx + C$

3. Linear Differential Equations

Equations of the form $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x.

- **Integrating Factor (I.F.):** $I.F. = e^{\int P dx}$
- **Solution Formula:** $y \cdot (I.F.) = \int (Q \cdot I.F.)dx + C$

Key Summary Table

Concept	**Key Tool/Formula**
Implicit Diff	Differentiate both sides; collect $\frac{dy}{dx}$
Local Maxima	$f'(c) = 0$ and $f''(c) < 0$
Marginal Revenue	$MR = \frac{dR}{dx}$
Integration by Parts	$uv - \int vdu$
Definite Integral	$F(b) - F(a)$
Order/Degree	Order = Highest derivative; Degree = Its power