Unit I: Numbers, Quantification and Numerical Applications

A. Modulo Arithmetic

Modulo arithmetic is a system of arithmetic for integers, where numbers "wrap around" upon reaching a certain value, known as the **modulus**.

1. Definition of Modulus of an Integer

In a mathematical sense, the modulus operation finds the remainder after division of one number by another. The sources illustrate this concept through the study of relations. If we consider a fixed positive integer n, then for any integers a and b, we say a is related to b if their difference $(a - b)$ is divisible by n.

Note from outside the sources: In standard modulo arithmetic notation, this is expressed as: $a \pmod{n} = r$ where r is the remainder such that $0 \leq r < n$. You may want to independently verify specific modular reduction rules for negative integers.

2. Arithmetic Operations using Modular Rules

Modular arithmetic allows for standard operations like addition, subtraction, and multiplication while maintaining the wrap-around property.

- **Addition Rule:** $(a + b) \pmod{n} = [(a \pmod{n}) + (b$ (mod
- **Multiplication Rule:** $(a \times b) \pmod{n} = [(a \pmod{n}) \times (b$ (mod

The sources demonstrate that such operations preserve the properties of **equivalence relations**, specifically **transitivity**. For instance, if $(a - b)$ is divisible by n and $(b - c)$ is divisible by n, then the sum $(a - b) + (b - c) = (a - c)$ is also divisible by n.

B. Congruence Modulo

Congruence modulo formalizes the relationship between two integers that leave the same remainder when divided by a modulus n.

1. Definition of Congruence Modulo

Two integers a and b are said to be **congruent modulo** n if n divides $(a - b)$. This is written as: $a \equiv b \pmod{n}$ This relationship is an **equivalence relation**, meaning it is:

- **Reflexive:** $a \equiv a \pmod{n}$.
- **Symmetric:** If $a \equiv b \pmod{n}$, then $b \equiv a \pmod{n}$.
- **Transitive:** If $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$.

2. Equivalence Classes

A modulus n divides the set of all integers into n mutually disjoint subsets called **equivalence classes**. For modulus 3, the integers are divided into three classes:

- $= \ldots, -6, -3, 0, 3, 6, \ldots$ (multiples of 3).
- $= \ldots, -5, -2, 1, 4, 7, \ldots$ (remainder 1).
- $= \ldots, -4, -1, 2, 5, 8, \ldots$ (remainder 2).

C. Alligation and Mixture

The **Rule of Alligation** is a numerical method used to determine the ratio in which two or more ingredients at different prices must be mixed to produce a mixture at a specific **mean price**.

Note from outside the sources: The specific "Rule of Alligation" formula is not explicitly detailed in the provided textbook excerpts. However, the logic of weighted averages is foundational to numerical applications. You may want to independently verify the following standard formula:
$$\frac{\text{Quantity of Cheaper}}{\text{Quantity of Dearer}} = \frac{\text{CP of Dearer} - \text{Mean Price}}{\text{Mean Price} - \text{CP of Cheaper}}$$

D. Numerical Problems (Mathematical Modeling)

Numerical applications often require translating real-life scenarios into mathematical structures like **matrices** to solve for multiple variables simultaneously.

1. Matrix Representation of Data

Matrices are powerful tools for organizing and solving numerical problems. For example, if three people possess different quantities of notebooks and pens, this can be represented in a rectangular array (matrix) where rows represent people and columns represent items.

2. Solving Real-Life Problems

The sources provide several instances where matrix algebra solves complex numerical applications:

- **Production Costs:** Calculating the total production of sports shoes across different factories and price categories using **matrix addition**.
- **Investment Strategy:** Dividing a trust fund of $30,000$ between two bonds paying 5% and 7% interest to achieve a specific annual return using **matrix multiplication**.
- **Election Expenditure:** Determining total spending in different cities by multiplying a "cost-per-contact" matrix by a "number-of-contacts" matrix.
- **Market Revenue:** Calculating total revenue and gross profit for products across different markets by multiplying quantity matrices by price/cost matrices.

E. Boats and Streams

Problems involving boats and streams are centered on the concept of **relative velocity** and **rate of change**.

1. Distinguishing Upstream and Downstream

The movement of a boat is influenced by the speed of the stream.

- **Downstream:** The boat moves in the same direction as the stream. The effective speed is the sum of the boat's speed in still water and the stream's speed.
- **Upstream:** The boat moves against the direction of the stream. The effective speed is the difference between the boat's speed and the stream's speed.

2. Equation Formulation

By applying the derivative $\frac{ds}{dt}$, we represent the rate of change of distance with respect to time. If u is the speed of the boat in still water and v is the speed of the stream:

- **Speed Downstream** $= u + v$
- **Speed Upstream** $= u - v$

These rates are used to form linear equations to solve for time or distance.

F. Pipes and Cisterns

Pipes and cisterns problems are numerical applications of work and time, often modeled using **rates of change**.

1. Determining Fill Time

If a pipe can fill a tank in x hours, the **rate of filling** is $\frac{1}{x}$ of the tank per hour.

- If two pipes fill a tank in x and y hours respectively, their combined rate is: $\text{Combined Rate} = \frac{1}{x} + \frac{1}{y}$
- The total time taken to fill the tank is the reciprocal of the combined rate: $\text{Total Time} = \frac{xy}{x+y}$

2. Emptying Pipes (Leakage)

If a pipe (or leak) empties the tank in z hours, its rate is considered negative ($-\frac{1}{z}$). The net rate of change of the volume of water in the tank is the algebraic sum of the rates of all pipes involved.

G. Races and Games

Races and games involve comparing the performance of players based on time, distance, and speed. These problems typically use the relationship between distance, constant speed, and time.

Note from outside the sources: Specific "Races and Games" logic (e.g., "A beats B by 10 meters") is not explicitly detailed in the excerpts. However, the sources discuss the comparison of ratios and proportionality to determine distances, similar to how ancient mathematicians calculated the distance of ships at sea. You may want to independently verify concepts of "Head Start" and "Dead Heat" in competitive racing.

H. Numerical Inequalities

Numerical inequalities describe the relationship between two values that are not equal, using symbols such as $<, >, \leq, \text{or} \geq$.

1. Basic Concepts

An inequality states that one expression is greater than or less than another. The sources use these to define **constraints** in real-world optimization problems. For example, a dealer with a storage limit of 60 pieces faces the inequality: $x + y \leq 60$ where x and y are the quantities of items.

2. Writing and Understanding Inequalities

Inequalities are used to define a **feasible region** on a coordinate plane.

- **Non-negative Constraints:** In most real-life numerical applications, variables cannot be negative ($x \geq 0, y \geq 0$).
- **Optimization:** Inequalities help determine the maximum or minimum values of an **objective function** (e.g., maximizing profit

$Z = 250x + 75y$) within a bounded or unbounded region.

- **Corner Point Method:** The optimal solution for a system of linear inequalities often occurs at the **vertices (corner points)** of the feasible region.

3. Properties of Inequalities

While standard arithmetic involves equalities, inequalities follow specific rules:

- The **absolute value** of a difference ($|a - b|$) is often used to ensure a non-negative result in numerical comparisons.
- Multiplying or dividing an inequality by a negative number reverses the inequality sign (**Note from outside the sources:** While the sources use inequalities for constraints, they do not explicitly list all algebraic properties of inequality manipulation. You may want to verify rules for multiplying by negative constants).

Summary of Key Formulae for CUET

Topic	Key Formula / Concept
Modulo	$a \equiv b \pmod{n} \Rightarrow n \text{ divides } (a -$
Matrix Addition	$c_{ij} = a_{ij} + b_{ij}$
Matrix Multiplication	$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$
Rate of Change	$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$
Area of Triangle	$\Delta = \frac{1}{2} \lvert x_1 \quad y_1 \quad 1 \; x_2 \quad y_2 \quad 1 \; x_3 \quad y_3$
LPP Objective	Max/Min $Z = ax + by$