1. Matrices

A. Concept and Notation

A **matrix** is defined as an ordered rectangular array of numbers or functions. The numbers or functions contained within this array are called its **elements** or **entries**. Matrices are typically denoted by capital letters (A, B, C).

In general, an $m \times n$ matrix is represented as: $A = [a_{ij}]m \times n, \quad 1 \le i \le m, \quad 1 \le j \le n \quad (i, j \in \mathbb{N})$. *Here,* aij is the element located in the i^{th} row and j^{th} column.

B. Order and Equality

- **Order:** A matrix with m rows and n columns is said to have the **order** $m \times n$ (read as "m by n"). The total number of elements in an $m \times n$ matrix is equal to the product mn.
- **Equality:** Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be **equal** ($A = B$) if and only if they satisfy two conditions:
  1. They have the same order.
  2. Every element of A is equal to its corresponding element in B ($a_{ij} = b_{ij}$ for all i, j).

C. Kinds of Matrices

The sources identify several specific structural types:

- **Column Matrix:** Contains only one column (order $m \times 1$).
- **Row Matrix:** Contains only one row (order $1 \times n$).
- **Square Matrix:** The number of rows equals the number of columns ($m = n$). For a square matrix of order n, the elements $a_{11}, a_{22}, \ldots, a_{nn}$ constitute its **diagonal**.
- **Diagonal Matrix:** A square matrix where all non-diagonal elements are zero ($a_{ij} = 0$ for $i \neq j$).
- **Scalar Matrix:** A diagonal matrix where all diagonal elements are equal to a constant k.
- **Identity Matrix:** A square matrix where diagonal elements are all 1 and non-diagonal elements are 0. It is denoted by I_n.
- **Zero Matrix:** A matrix where all elements are zero. It is denoted by O.

D. Operations on Matrices

1. Addition and Subtraction

The sum of two matrices A and B is defined only if they are of the **same order**. The result is a matrix where each element is the sum of the corresponding elements of A and B: $C = [a_{ij} + b_{ij}]$.

- **Properties:** Addition is both **commutative** ($A + B = B + A$) and **associative** ($(A + B) + C = A + (B + C)$).
- **Identity and Inverse:** The zero matrix O is the additive identity ($A + O = A$). The negative of a matrix, denoted $-A$, is the additive inverse such that $A + (-A) = O$.

2. Scalar Multiplication

If k is a scalar and A is a matrix, then kA is obtained by multiplying every element of A by k.

- **Properties:** $k(A + B) = kA + kB$ and $(k + l)A = kA + lA$.

3. Matrix Multiplication

The product AB is defined **only if the number of columns in A equals the number of rows in B**. If A is $m \times n$ and B is $n \times p$, the resulting matrix C is of order $m \times p$. The element c_{ik} is calculated as:

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}.$$

Critical Properties for CUET:

- **Non-Commutativity:** Generally, $AB \neq BA$. Even if both products are defined, they may have different orders or different values.
- **Diagonal Exception:** Multiplication of diagonal matrices of the same order is commutative.

- **Zero Product Rule:** Unlike real numbers, the product of two non-zero matrices can be a zero matrix.
  - *Example (Order 2):* If $A = \begin{bmatrix} 0 & -1 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 0 & 0 \end{bmatrix}$, then $AB = O$ despite $A, B \neq O$.
- **Multiplicative Identity:** For every square matrix A, $IA = AI = A$.

E. Transpose, Symmetric, and Skew-Symmetric Matrices

- **Transpose (A' or A^T):** Formed by interchanging the rows and columns of A.
  - **Properties:** $(A')' = A$, $(kA)' = kA'$, $(A+B)' = A' + B'$, and $(AB)' = B'A'$ (Reversal Law).
- **Symmetric Matrix:** A square matrix A is symmetric if $A' = A$.
- **Skew-Symmetric Matrix:** A square matrix A is skew-symmetric if $A' = -A$.
  - **Key Property:** All diagonal elements of a skew-symmetric matrix must be zero ($a_{ii} = 0$).
- **Decomposition Theorem:** Any square matrix can be expressed uniquely as the sum of a symmetric matrix $\frac{1}{2}(A + A')$ and a skew-symmetric matrix $\frac{1}{2}(A - A')$.

F. Invertible Matrices

A square matrix A of order m is **invertible** if there exists a square matrix B of the same order such that: $AB = BA = I$. In this case, B is called the **inverse** of A (denoted A^{-1}).

- **Uniqueness Proof:** If B and C are both inverses of A, then $B = BI = B(AC) = (BA)C = IC = C$. Hence, the inverse is unique.
- **Product Inverse:** If A and B are invertible, then $(AB)^{-1} = B^{-1}A^{-1}$.

---

2. Determinants

A. Concept and Evaluation

A **determinant** is a unique number (real or complex) associated with every square matrix. It is denoted by $|A|$, $\det A$, or Δ.

- **Order 2 Expansion:** For $A = \begin{bmatrix} a_{11} & a_{12} a_{21} & a_{22} \end{bmatrix}$, $|A| = a_{11}a_{22} - a_{21}a_{12}$.
- **Order 3 Expansion:** Expanded along any row or column. Expansion along the first row: $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$. Where A_{ij} is the cofactor. For easier calculation, expand along the row/column with the maximum number of zeros.

B. Minors and Cofactors

- **Minor (M_{ij}):** The determinant of the submatrix obtained by deleting the i^{th} row and j^{th} column.
- **Cofactor (A_{ij}):** Defined as $A_{ij} = (-1)^{i+j} M_{ij}$.

- **Key Identity:** The sum of the products of elements of any row/column with their corresponding cofactors equals $|A|$. If elements are multiplied by cofactors of a *different* row/column, the sum is zero.

C. Applications of Determinants

- **Area of a Triangle:** For vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, the area Δ is: $\Delta = \frac{1}{2} |x_1 \quad y_1 \quad 1\ x_2 \quad y_2 \quad 1\ x_3 \quad y_3 \quad 1|$. Always take the absolute value as area must be positive. If the area is zero, the points are **collinear**.

D. Adjoint and Inverse of a Square Matrix

- **Adjoint ($\text{adj } A$):** The transpose of the matrix of cofactors of A.
  - **Property:** $A(\text{adj } A) = (\text{adj } A)A = |A|I$.
- **Singularity:** A matrix is **singular** if $|A| = 0$ and **non-singular** if $|A| \neq 0$.
- **Inverse Formula:** A square matrix A is invertible if and only if it is non-singular. The formula is: $A^{-1} = \frac{1}{|A|}\text{adj } A$.

E. Solving Systems of Linear Equations

A system of linear equations can be written as $AX = B$.

- **Matrix Method:** If $|A| \neq 0$, the system has a **unique solution** given by: $X = A^{-1}B$.
- **Consistency and Inconsistency:**
  1. **Consistent:** One or more solutions exist. This occurs if $|A| \neq 0$ (unique).
  2. **Inconsistent:** No solution exists. This occurs if $|A| = 0$ and $(\text{adj } A)B \neq O$.
  3. **Infinite Solutions:** May occur if $|A| = 0$ and $(\text{adj } A)B = O$.

Summary of Formulae for Quick Review

Concept	Formula / Rule
Matrix Order	$m \times n$ (Rows $\times$ Columns)
Skew-Symmetric	$A^T = -A$ and $a_{ii} = 0$
Product Rule	$(AB)^T = B^T A^T$
Inverse Formula	$A^{-1} = \frac{1}{\det A}\text{adj } A$
Area of Triangle	$$\frac{1}{2}
Adjoint Property	$$
Matrix Method	$X = A^{-1}B$