Short Notes of Mathematics for CUET(B2)

UNIT VIII: LINEAR PROGRAMMING

A. Introduction and Related Terminology

To master Linear Programming, students must first become familiar with its specific vocabulary:

1. **Objective Function:** A linear function of the form $Z = ax + by$, where a and b are constants, which must be maximised or minimised.
2. **Decision Variables:** The variables x and y in the objective function whose values we need to determine to achieve the optimal result.
3. **Constraints:** These are the linear inequalities, equations, or restrictions placed on the decision variables. For example, in a furniture business, storage space might be represented as $x + y \leq 60$.
4. **Non-negative Restrictions:** In most real-world scenarios, the variables cannot be negative (e.g., you cannot produce a negative number of chairs). These are written as $x \geq 0$ and $y \geq 0$.
5. **Optimisation Problem:** A general class of problems that seeks the "best" (maximum or minimum) outcome within given limits. Linear Programming Problems (LPP) are a specific type of optimisation problem where all relationships are linear.

B. Mathematical Formulation of LPP

Formulating an LPP involves translating a real-life situation into mathematical expressions.

Step-by-Step Formulation

The sources illustrate this using the example of a furniture dealer with $Rs\ 50,000$ to invest and storage for at most 60 pieces. Tables cost $Rs\ 2500$ and chairs cost $Rs\ 500$, with profits of $Rs\ 250$ and $Rs\ 75$ respectively.

1. **Identify Variables:** Let x be the number of tables and y be the number of chairs.
2. **State Non-negative Constraints:** $x \geq 0, \quad y \geq 0$.
3. **Write the Constraints:**
   - **Investment constraint:** $2500x + 500y \leq 50000$, which simplifies to $5x + y \leq 100$.
   - **Storage constraint:** $x + y \leq 60$.
4. **Define the Objective Function:** The dealer wants to maximise profit Z: $\text{Maximise } Z = 250x + 75y$.

C. Different Types of Linear Programming Problems

The sources highlight several classic types of LPP based on their practical applications:

1. **Manufacturing Problems:** These involve determining the number of units of different products a company should produce to maximise profit, given constraints on manpower, machine hours, and raw materials.
2. **Diet Problems: (Note from outside the sources)** These involve determining the minimum cost of food items that meet specific nutritional requirements. The historical notes mention that economist G. Stigler first described this in 1945.
3. **Transportation Problems:** These involve determining a transportation schedule to find the cheapest way of shipping a product from plants/factories to different markets.
4. **Investment Strategy:** Deciding how to divide a fixed sum of money between different bonds or assets to maximise annual return.

D. Graphical Method of Solution

For problems involving only **two variables**, the graphical method is the most intuitive approach. It involves graphing the system of linear inequalities on a coordinate plane.

The Corner Point Method

This is a fundamental technique for finding the optimal solution. The logic is supported by two key theorems:

- **Theorem 1:** The optimal value (maximum or minimum) of an objective function must occur at a **corner point (vertex)** of the feasible region.
- **Theorem 2:** If the feasible region is **bounded**, the objective function always has both a maximum and a minimum value, both occurring at corner points.

Procedure for Graphical Solution:

1. **Graph the Constraints:** Treat each inequality as an equation to draw a line. Use test points (like $0, 0$) to determine which side of the line represents the inequality.
2. **Identify the Feasible Region:** Find the region where all shaded areas overlap.
3. **Find Corner Points:** Identify the vertices of this region either by inspection or by solving the equations of the intersecting lines.
4. **Evaluate Z:** Substitute the coordinates of each corner point into the objective function.
5. **Select the Optimal Value:** The largest value represents the maximum, and the smallest represents the minimum.

E. Feasible and Infeasible Regions

1. Feasible Region

The **feasible region** is the common region determined by all constraints, including the non-negative restrictions. It is always a **convex region**.

- **Bounded Region:** A feasible region is bounded if it can be enclosed within a circle. Bounded regions are guaranteed to have both a maximum and a minimum value.
- **Unbounded Region:** A region that extends indefinitely in one or more directions. In this case, an optimal value may or may not exist.

2. Infeasible Region

The region outside the feasible area is the **infeasible region**. Points in this region do not satisfy at least one of the constraints. If there is no overlap between the constraints, the problem has **no feasible region** and, consequently, no solution.

F. Feasible, Infeasible, and Optimal Solutions

1. Solutions Defined

- **Feasible Solution:** Any point within or on the boundary of the feasible region. These points satisfy all the constraints.
- **Infeasible Solution:** Any point outside the feasible region.
- **Optimal Feasible Solution:** The specific point in the feasible region that gives the maximum (or minimum) value of the objective function.

2. Finding the Optimal Solution

In a **bounded** region, the process is straightforward: calculate Z at all corner points and compare.

Handling Unbounded Regions: If the region is unbounded, a value M found at a corner point is the maximum **only if** the open half-plane determined by $ax + by > M$ has no points in common with the feasible region. If it does have common points, the maximum value does not exist. A similar check is used for the minimum value ($ax + by < m$).

3. Multiple Optimal Solutions

Sometimes, two different corner points (e.g., points C and D) may yield the same maximum value. In such cases, **every point on the line segment** joining these two corner points is also an optimal solution.

Summary for CUET Preparation

Term	Mathematical Definition
Objective Function	$Z = ax + by$
Decision Variables	$x, y \geq 0$
Optimal Point	Occurs at a **Vertex** of the feasible region
Bounded Region	Enclosed region; Max and Min always exist
No Solution	Occurs when no common feasible region exists
Simplex Method	**(Note from outside the sources)** An iterative procedure used for complex LPP with many variables, suggested by G.B. Dantzig in 1947.