Short Notes of Mathematics for CUET(1B)
UNIT IV: VECTORS AND THREE-DIMENSIONAL GEOMETRY

1. Vectors: Fundamentals and Operations

A. Vectors and Scalars

In mathematical and physical contexts, quantities are classified into two types:

- **Scalars:** Quantities involving only magnitude, represented by a real number. Examples include height (e.g., $1.6\ \text{meters}$), mass, time, distance, speed, area, and temperature.
- **Vectors:** Quantities involving both magnitude and a specific direction. Examples include displacement, velocity, acceleration, force, weight, momentum, and electric field intensity.

The word "vector" is derived from the Latin *vectus*, meaning "to carry".

B. Magnitude and Direction of a Vector

A **directed line segment** represents a vector. If a line l is restricted to a segment AB with a prescribed direction, we obtain a vector $\vec{AB}$.

- **Initial Point:** The starting point A.
- **Terminal Point:** The endpoint B.
- **Magnitude (Length):** The distance between the initial and terminal points, denoted as $|\vec{AB}|$ or a. Length is never negative.

C. Position Vector, Direction Cosines, and Ratios

1. Position Vector

In a three-dimensional rectangular coordinate system, the **position vector** of a point $P(x, y, z)$ with respect to the origin $O(0, 0, 0)$ is the vector $\vec{OP}$. The magnitude is calculated using the distance formula: $|\vec{OP}| = \sqrt{x^2 + y^2 + z^2}$.

2. Direction Cosines and Ratios

- **Direction Angles:** The angles α, β, γ that a position vector makes with the positive $x, y,$ and z axes.
- **Direction Cosines (DCs):** The values $l = \cos\alpha, m = \cos\beta,$ and $n = \cos\gamma$. A key property is: $l^2 + m^2 + n^2 = 1$.
- **Direction Ratios (DRs):** Numbers a, b, c that are proportional to the direction cosines. For a vector $\vec{r}$, these are simply its scalar components $x, y,$ and z.
- **Relationship:** $l = \frac{a}{|\vec{r}|}, m = \frac{b}{|\vec{r}|}, n = \frac{c}{|\vec{r}|}$.

D. Types of Vectors

The sources identify several fundamental vector types:

- **Zero (Null) Vector:** Initial and terminal points coincide. Magnitude is 0; direction is indeterminate.
- **Unit Vector:** Magnitude is exactly 1 unit. The unit vector in the direction of $\vec{a}$ is denoted $\hat{a}$ and calculated as: $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.
- **Coinitial Vectors:** Vectors having the same initial point.
- **Collinear (Parallel) Vectors:** Vectors parallel to the same line, regardless of magnitude or sense of direction.
- **Equal Vectors:** Vectors with the same magnitude and direction, regardless of initial point position.
- **Negative of a Vector:** A vector with the same magnitude as $\vec{a}$ but exactly the opposite direction, written as $-\vec{a}$.

E. Components and Vector Addition

1. Components of a Vector

Vectors are often expressed in terms of mutually perpendicular unit vectors $\hat{i}, \hat{j},$ and $\hat{k}$ along the $x, y,$ and z axes. For a point $P(x, y, z)$, the vector is: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. x, y, z are scalar components, while $x\hat{i}, y\hat{j}, z\hat{k}$ are vector components.

2. Addition and Multiplication

- **Triangle Law:** If vectors $\vec{a}$ and $\vec{b}$ are sides of a triangle in order, their sum is the third side $\vec{AC} = \vec{AB} + \vec{BC}$.
- **Parallelogram Law:** If $\vec{a}$ and $\vec{b}$ are adjacent sides of a parallelogram, their sum is the diagonal passing through their common point.
- **Scalar Multiplication:** Multiplying vector $\vec{a}$ by scalar λ results in $\lambda\vec{a}$. Magnitude becomes $|\lambda||\vec{a}|$. Direction remains the same if $\lambda > 0$ and reverses if $\lambda < 0$.

3. Section Formula

The position vector $\vec{r}$ of a point R dividing the line segment PQ (with position vectors $\vec{a}$ and $\vec{b}$) in the ratio $m : n$:

- **Internal Division:** $\vec{r} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$.
- **External Division:** $\vec{r} = \dfrac{m\vec{b} - n\vec{a}}{m-n}$.

2. Product of Vectors

A. Scalar (Dot) Product

The scalar product of two vectors $\vec{a}$ and $\vec{b}$ is a real number: $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$ Where θ is the angle between them ($0 \le \theta \le \pi$).

Key Observations:

- **Perpendicularity:** $\vec{a} \cdot \vec{b} = 0$ if and only if $\vec{a} \perp \vec{b}$.
- **Commutativity:** $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.
- **Components:** If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then: $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.

Projection of a Vector: The projection of $\vec{a}$ on $\vec{b}$ is given by: $\text{Projection} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \vec{a} \cdot \hat{b}$.

B. Vector (Cross) Product

The vector product results in a vector perpendicular to both $\vec{a}$ and $\vec{b}$: $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\hat{n}$ Where $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ following the right-hand rule.

Key Observations:

- **Parallelism:** $\vec{a} \times \vec{b} = \vec{0}$ if and only if $\vec{a} \parallel \vec{b}$.
- **Non-Commutative:** $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$.
- **Geometric Interpretation:**
  - **Area of Triangle:** $\frac{1}{2}|\vec{a} \times \vec{b}|$.
  - **Area of Parallelogram:** $|\vec{a} \times \vec{b}|$.
- **Determinant Form:** $\vec{a} \times \vec{b} = |\hat{i} \quad \hat{j} \quad \hat{k} \; a_1 \quad a_2 \quad a_3 \; b_1 \quad b_2 \quad b_3|$.

3. Three-Dimensional Geometry

This section applies vector algebra to define the properties of lines in space.

A. Direction Cosines and Ratios of a Line

A line joining points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ has direction ratios: $a = x_2 - x_1, b = y_2 - y_1, c = z_2 - z_1$. The direction cosines are: $l = \frac{x_2 - x_1}{PQ}, m = \frac{y_2 - y_1}{PQ}, n = \frac{z_2 - z_1}{PQ}$
Where PQ is the distance between the points.

B. Equation of a Line

1. Through a point parallel to a vector

- **Vector Equation:** $\vec{r} = \vec{a} + \lambda\vec{b}$.
- **Cartesian Equation:** $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$.

2. Through two given points

- **Vector Equation:** $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$.
- **Cartesian Equation:** $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$.

C. Angle Between Two Lines

For two lines with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2, the acute angle θ is:

$$\cos\theta = \left| \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} \right|.$$

- **Perpendicular lines:** $a_1a_2 + b_1b_2 + c_1c_2 = 0$.
- **Parallel lines:** $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

D. Shortest Distance Between Two Lines

1. Skew Lines

Skew lines are lines in space that are neither parallel nor intersecting and lie in different planes. The shortest distance d between lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$: $d = \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$.

2. Parallel Lines

For two parallel lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}$: $d = \frac{|\vec{b} \times (\vec{a}_2 - \vec{a}_1)|}{|\vec{b}|}$.

Summary Table for CUET Preparation

Concept	Key Formula
Dot Product	$$ \vec{a} \cdot \vec{b} =
Cross Product	$$ \vec{a} \times \vec{b} =
Line (Vector)	$\vec{r} = \vec{a} + \lambda\vec{b}$
Angle between Lines	$$ \cos \theta = \frac{
Shortest Distance (Skew)	$$ \frac{
Area of Parallelogram	$$$$