Short Notes of Mathematics for CUET(1B)

UNIT I: RELATIONS AND FUNCTIONS

1. Integration as the Inverse Process of Differentiation

Integration is the process of finding the original function when its derivative is given. If differentiating a function F(x) gives f(x), then integrating f(x) returns F(x).

- **Fundamental Definition:** If $\frac{d}{dx}[F(x)] = f(x)$, then $\int f(x)dx = F(x) + C$
- **Constant of Integration (C):** Represents the family of parallel curves. Since the derivative of any constant is zero, an indefinite integral must always include + C.

2. Key Methods of Integration

A. Integration by Substitution

Used when an integrand contains a function and its derivative.

- **Rule:** If you have $\int f(g(x))g'(x)dx$, substitute $g(x) = t$.
- **Differentiation:** This implies $g'(x)dx = dt$.
- **New Integral:** The integral simplifies to $\int f(t)dt$, which is evaluated and then substituted back to x.
- **Standard Form:** $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$

B. Integration by Partial Fractions

Used to integrate rational functions of the form $\frac{P(x)}{Q(x)}$ where P(x) and Q(x) are polynomials, and the degree of P(x) is strictly less than Q(x) (proper fraction).

Standard Decompositions:

- **Distinct Linear Factors:** $\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$
- **Repeated Linear Factors:** $\frac{1}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
- **Irreducible Quadratic Factor:** $\frac{1}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$

C. Integration by Parts

Used for the product of two distinct functions.

- **Formula:** $\int u \cdot v\, dx = u \int v\, dx - \int \left[\frac{du}{dx} \int v\, dx \right] dx$
- **ILATE Rule (Choosing u):** Choose the first function (u) based on this priority order:
  1. **I**nverse Trigonometric ($\sin^{-1}x$, $\tan^{-1}x$)
  2. **L**ogarithmic ($\ln x$)
  3. **A**lgebraic (x^2, polynomials)
  4. **T**rigonometric ($\sin x$, $\sec^2 x$)
  5. **E**xponential (e^x, a^x)

3. Evaluation of Standard Simple Integrals

These are the foundational formulas required for CUET. Memorization of these forms is essential.

Forms without Square Roots:

- $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$
- $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C$
- $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right| + C$

Forms with Square Roots in the Denominator:

- $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left|x + \sqrt{x^2 + a^2}\right| + C$
- $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left|x + \sqrt{x^2 - a^2}\right| + C$
- $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

4. Integration Involving Quadratic Expressions

Type I: General Quadratics

Forms: $\int \frac{dx}{ax^2 + bx + c}$ and $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$

Methodology (Completing the Square):

1. Factor out a to make the coefficient of x^2 unity: $\frac{1}{a} \int \frac{dx}{x^2 + \frac{b}{a}x + \frac{c}{a}}$
2. Add and subtract the square of half the coefficient of x: $\left(\frac{b}{2a}\right)^2$

3. Rewrite the denominator as a perfect square plus or minus a constant: $\left(x + \frac{b}{2a}\right)^2 \pm k^2$
4. Apply the relevant standard integral formula from Section 3.

Type II: Linear / Quadratic Combinations

Forms: $\int \frac{(px + q)dx}{ax^2 + bx + c}$ and $\int \frac{(px + q)dx}{\sqrt{ax^2 + bx + c}}$

Methodology:

1. Express the linear numerator as a derivative of the quadratic denominator plus a constant:
   $px + q = A \cdot \frac{d}{dx}(ax^2 + bx + c) + B$
2. Expand the right side: $px + q = A(2ax + b) + B$
3. Equate the coefficients of x and the constant terms to find the values of A and B.
4. Split the integral into two separate integrals:
   - **Integral 1:** Involves A(2ax+b) in the numerator. Solve using the substitution $t = ax^2 + bx + c$.
   - **Integral 2:** Involves the constant B in the numerator. Solve using the "Completing the Square" method (Type I).

5. Integrals of Square Roots of Quadratics (Numerator)

These formulas frequently appear in CUET coordinate geometry applications (like finding the area of circles and ellipses).

Standard Forms:

- $\int \sqrt{a^2 - x^2} dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$
- $\int \sqrt{x^2 + a^2} dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln\left|x + \sqrt{x^2 + a^2}\right| + C$
- $\int \sqrt{x^2 - a^2} dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\left|x + \sqrt{x^2 - a^2}\right| + C$

General Form: $\int \sqrt{ax^2 + bx + c} dx$

- **Methodology:** Use the exact same "Completing the Square" technique described in Section 4 (Type I) to transform the quadratic $ax^2 + bx + c$ into the form $A^2 - X^2$, $X^2 + A^2$, or $X^2 - A^2$. Then apply the corresponding standard formula above.